Cubic, Square and Square Root Transformation of a Gamma Distribution: A Comparative Analysis

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Abstract
This work is on the destruction of Weibull under cubic, square and square root transformation. The data for the analysis were simulated. Based on the analysis, it was discovered that the mean and variance of cubic and square root transformation are normally distributed while the mean of square even before and after transformation are unity and the variance of the transformed data is greater than the original data. Hence, one should be very careful in applying square root transformation to a data set where the multiplicative error under model is a suitable representation and where the error component has a Weibull distribution. This implies that the transformation increases the error variance.

Key words: *Weibull distribution, Gamma Distribution, Transformation of Data, Probability Density Function and Cumulative Density Function.*

1.0 GAMMA DISTRIBUTION

Gamma distribution is sometimes referred to as distributions of waiting time. It provides a good model for waiting time and reliability theory. It is used to model workers income since income is a non-negative random variable. In other words, it's a model. The gamma distribution like the lognormal distribution is an alternative to consider for ecological variables that seem to be highly skewed. If the random variable Y is gamma distributed with parameter α and β. Then the likelihood of Y is $P(Y)=\frac{\beta^{a}}{\Gamma(a)}\gamma^{a-1}e^{-\beta\gamma}$ where the gamma functions $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$ as in lognormal distribution Y and the parameters α and β must be positive.

The gamma distribution function has three different types 1-, 2-, 3- parameter gamma distribution if the continuous random variable x fits the probability density function of

$$f(x)=\frac{1}{\Gamma(\alpha)}x\alpha^{-1}e^{-x}, x\geq 0 \qquad (1)$$

It is said that the variable x is 1- parameter gamma distributed, with shape parameter α. In equation (1), $\Gamma(\alpha)$ the incomplete gamma function is given by

$$\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx \qquad (2)$$

The distribution function has a form of the simple exponential distribution in the case of $\alpha=1$; if x in equation (2) is replaced by x/β, where β is the scale parameter, then 2-parameter gamma distribution is obtained as

$$F(x)-\frac{1}{\beta^{\alpha}\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta}; x\geq 0 \qquad (3)$$

which returns to the 1- parameter gamma distribution for $\beta = 2.$ If x is replaced by $(x-r)/\beta$, where Γ is the location parameter, then the 3- parameter gamma distribution is obtained as

$$F(x) - \frac{1}{\beta^{\alpha}\Gamma(\alpha)}(x-r)^{\alpha-1}e^{-(x-r)/\beta}; x \geq r \quad (4)$$

Hence, understanding the transformation of x variables in a two parameter gamma distribution by transformation of the square, square root and cubic will help us to understand the characteristics of gamma distribution.

The objectives of this study is to find if there is any difference in cubic, square and square root transformation of gamma distribution when compared to the main distributions, to find out if the means and variances of the transformed distribution are different from the main distribution and to find out the distributions that are more efficient by using the ratio of the mean and variance of the transformation when compare to the main distribution. This work will strengthen readers of this work to know how to model non-negative random variable.

The gamma distribution is characterized with the following properties: **SKEWNESS**

The Skewness which is equal to $\frac{2}{\sqrt{k}}$, depends only on the shape parameter (k) and approaches a normal distribution when k is large (approximately when k = 10).

MEDIAN CALCULATION

Unlike the mode and mean which have readily formulae based on the parameters, the median does not have an easy closed form equation. The median for this distribution is defined as the value ν such that $\frac{1}{\Gamma(k)\theta^{k}}\int_0^{\nu} x^{k-1}\ell^{-\frac{x}{\theta}}dx = \frac{1}{2}$

A rigorous treatment of the problem of determining an asymptotic expansion and bounds for the median of the gamma distribution was handled first by Ched and Rubin, who proved that $m - \frac{1}{3} < \lambda(m) < m,$ Where $\lambda(m)$ denotes the median of the gamma (m, 1) distribution (Fraser D.A.S 1967).

1.1 STATEMENT OF HYPOTHESIS

a. **H_{01}:** There is no significant difference between cubic transformation distribution and the main distribution.
b. **H_{02}:** There is no significant difference between square transformation distribution and the main distribution
c. **H_{03}:** There is no significant difference between square root transformation distribution and the main distribution.

1.2 Methods of Estimating Gamma Distribution

Observing n independent data points $x = (x_1, x_2, x_3, ..., x_n)$ from the same density $\theta = (\alpha, \beta)$. $P(x/\alpha, \beta) = G\alpha(x:\alpha, \beta) = \frac{x^{a-1}}{\Gamma(\alpha)\beta^{\alpha}}\exp\left(\frac{-x}{\beta}\right)$

Maximum Likelihood Estimator: Let $x_1, x_2, \ldots, x_n$ are independent and identically distributed random variables taken from gamma distribution, then the probability density function is given by $f(x:\alpha;\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{-1} \ell^{-\frac{x}{\beta}}; 0 < x < \infty; \alpha, \beta > 0$

The likelihood function is given by $L(x;\alpha,\beta) = \frac{n}{\prod_{r\backslash 1}} \frac{1}{\Gamma(\alpha)\beta\alpha} x^{-1} \ell^{-\sum_{r=1}^{n}\frac{X_i}{\beta}}$

The log likelihood function is

$$Log\{L(x;\alpha,\beta)\} = -n\log\Gamma(\alpha) - n\alpha\log\beta + (\alpha-1)\sum_{i=1}^{n}\log x_i - \frac{1}{\beta}\sum_{i=1}^{n} X_i$$

The likelihood equations are given by

$$\frac{\partial\{Log(L)\}}{\partial\alpha} = \frac{-n\Gamma^{I}(\alpha)}{\Gamma(\alpha)} n\log\beta + \sum_{i=1}^{n}\log X_i = 0 \quad \cdots \quad (5)$$

While, $\frac{\partial\{Log(L)\}}{\partial\beta} = n\frac{\alpha}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n} X_i = 0$ . . . (6)

Solving for β we have $\hat{\beta} = \frac{\hat{X}}{\alpha}$, substituting the value of β in equation (5) we get

$\frac{\Gamma^{I}(\alpha)}{(\alpha)} - \log\alpha = \frac{1}{n}\sum_{i=1}^{n}\log X_i - \log(\overline{X})$. This converges in about four iterations.

Methods of Moment Estimator: Let $x_1, x_2, \ldots, x_n$ be a random sample from gamma (α,β) distribution where both the parameters are unknown. The problem is to get the estimator for both α and β.

The first two moments about the origin are given by the following:

$\mu^{I}{}_1 = E(x) = \alpha\beta$ . . . (8)

$N_2{}^{I} E(x^2) = \alpha(\alpha+1)\beta^2$ . . . (9)

Then the sample moments are obtained as $\mu^{I}{}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\mu^{I}{}_2 = \frac{1}{n}\sum_{i=1}^{n} X^2{}_i$

So, using method of moments, we have that $\mu_1^{I} - N_1^{I} \Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i = \alpha\beta \Rightarrow \hat{\alpha} = \frac{\overline{x}}{\beta}$ and

$$\mu_2^{I} - N^{I}{}_2 \Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \alpha(\alpha+1)\beta^2 = \frac{\overline{x}}{\beta}\left(\frac{\overline{x}}{\beta}+1\right)\beta^2$$

Solving for β we have $\frac{1}{n}\sum_{i=1}^{n} X^2{}_i = \overline{X}(\overline{X}+\beta) \Rightarrow \beta\overline{x} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\overline{x})^2$

Finally we get $\hat{\beta} = \frac{S^2}{n\overline{x}}$ where $S^2 = \sum_{i=1}^{n}(x_i - \overline{x})^2$

The method of moment estimators of α and β are $\hat{\alpha} = \frac{\overline{x}}{\beta}$ and $\hat{\beta} = \frac{S^2}{n\overline{x}}$

Based on the fact that our interest is on the probability distributions or densities of functions of one or more random variables. Suppose we have a two sets of random variables, x_1, x_2, x_3,...x_n, with a known joint probability and or density function, we may want to know the distribution of some function of these random variables $\gamma = \phi(x_1, x_2, x_3, ..., x_n)$ where realized values of Y will be related to realized values of the x's as follows: $\gamma = \phi(x_1, x_2, x_3, ..., x_n)$.

2.0 Methods of Transformation

In distribution theory, there are three methods of transforming random variables, these are **Distribution Function Theory:** In this method, the region in x_1, x_2, x_3, ...,x_n space such that $\phi(x_1, x_2, x_3, ..., x_n) \leq 0$. We can find the probability that $\phi(x_1, x_2, x_3, ..., x_n) \leq \phi$, that is, $P\{\phi(x_1, x_2, x_3, ..., x_n) \leq \phi\}$ by integrating the density function $F(x_1, x_2, x_3, ..., x_n)$ over this region. In this case, $F(\phi)$ is just $P[\phi \leq \phi]$. Once $F(\phi)$ is obtained, the density can be obtained by integration.

Method of Transformation (Inverse Mappings): If the density function of X is unknown. In addition, suppose that the function of $y = \phi(x)$ is differentiable and monstonic for values within its range for which the density $F(x) \neq 0$. This means that we can solve the equation $y = \phi(x)$ for X as a function of Y. We can then use this inverse mapping to find the density function of Y. We can do a similar thing when there is more than one variable X and then there is more than one mapping $\phi(\ )$. (BALLINGSLEY, 1995)

Methods of Moment Generating Function: This model states that if two random variables have identical moment generating functions, they possess the same probability distribution. The procedure is to find the moment garneting function for ϕ and then compare it to any and all known ones to see if there is a match. This is most commonly done to see if a distribution approaches to normal distribution as the sample size goes to infinity.

3.0 Research Methodology

The work is to analyze the ratio work, to analyze the original mean and transformed mean, original variance and transformed variance. The data is a montecalous stimulated data that consists of 150 observations and 150 transformation into cubic, square and square root of gamma distribution.

3.1 Method of Analysis

The ratio method would apply to analyze the data to determine if there is any difference between the observations of data. In this work, we would compare the means $(E\xi)$ and $E(\xi *)$ and the variance $Var(\xi)$ and $Var(\xi *)$ by measuring the percentage increase in them to know if there is any other transformation; that is we would obtain the ratios $\frac{E(\xi *)}{E(\xi)}$ and $\frac{Var(\xi *)}{Var\xi}$.

For this purpose, we would assume $\theta = 1.0$ which means that if the mean is less than one, the transformed mean is better, if the mean is higher than the one, the original mean is better and if both are the same, the mean is normal. (Ohakwe et al (2013). The ratio is just a comparison between two different things.

4.0 PRESENTATION OF DATA

Table 4.1: Distribution of Gamma Simulated Data

Gamma	CUBE ROOT	SQUARE ROOT	SQUARE
48.6418	3.6504	6.9744	2366.0260
110.8092	4.8031	10.5266	12278.6690
29.9465	3.1054	5.4723	896.7952
37.2675	3.3402	6.1047	1388.8678
135.5361	5.1367	11.6420	18370.0428
118.7873	4.9158	10.8990	14110.4198
31.7344	3.1660	5.6333	1007.0733
28.8697	3.0677	5.3731	833.4584
28.6716	3.0607	5.3546	822.0585
42.9027	3.5008	6.5500	1840.6375
38.0140	3.3624	6.1656	1445.0675
19.4401	2.6888	4.4091	377.9170
15.9896	2.5193	3.9987	255.6683
91.5032	4.5062	9.5657	8372.8360
53.0974	3.7586	7.2868	2819.3339
39.9046	3.4172	6.3170	1592.3753
59.7596	3.9096	7.7304	3571.2039
52.0615	3.7340	7.2154	2710.4026
96.0709	4.5800	9.8016	9229.6245
64.4463	4.0093	8.0278	4153.3232
16.6041	2.5512	4.0748	275.6965
39.2884	3.3995	6.2680	1543.5768
169.2304	5.5313	13.0089	28638.9279
86.0425	4.4147	9.2759	7403.3081
45.6345	3.5735	6.7553	2082.5043
78.1656	4.2757	8.8411	6109.8538
42.4628	3.4887	6.5164	1803.0911
53.7531	3.7740	7.3317	2889.3952
28.4764	3.0537	5.3363	810.9030
34.4533	3.2539	5.8697	1187.0302
55.5924	3.8166	7.4560	3090.5174
53.2087	3.7612	7.2944	2831.1668
65.6412	4.0339	8.1019	4308.7645
77.4420	4.2624	8.8001	5997.2605
43.5416	3.5180	6.5986	1895.8718
25.8416	2.9565	5.0835	667.7874
55.9053	3.8237	7.4770	3125.3991
52.9727	3.7556	7.2782	2806.1020
49.6630	3.6757	7.0472	2466.4125

48.8722	3.6561	6.9909	2388.4917
43.7972	3.5249	6.6179	1918.1927
31.2797	3.1508	5.5928	978.4174
53.3501	3.7645	7.3041	2846.2363
107.7419	4.7584	10.3799	11608.3100
78.3061	4.2782	8.8491	6131.8398
16.5689	2.5494	4.0705	274.5270
62.4870	3.9682	7.9049	3904.6265
65.4994	4.0310	8.0932	4290.1709
52.3893	3.7418	7.2380	2744.6418
24.6824	2.9116	4.9681	609.2232
42.1032	3.4789	6.4887	1772.6776
41.8546	3.4720	6.4695	1751.8089
18.0761	2.6244	4.2516	326.7462
70.3479	4.1281	8.3874	4948.8219
37.8077	3.3563	6.1488	1429.4199
36.0491	3.3034	6.0041	1299.5407
58.5219	3.8825	7.6500	3424.8186
43.5736	3.5189	6.6010	1898.6616
52.2050	3.7374	7.2253	2725.3670
24.6414	2.9100	4.9640	607.1986
26.2963	2.9737	5.1280	691.4941
62.8284	3.9754	7.9264	3947.4051
45.4929	3.5698	6.7448	2069.6006
31.8617	3.1702	5.6446	1015.1697
81.9507	4.3436	9.0527	6715.9231
70.6744	4.1345	8.4068	4994.8720
93.3335	4.5361	9.6609	8711.1476
58.5198	3.8824	7.6498	3424.5689
99.2334	4.6297	9.9616	9847.2614
43.2567	3.5104	6.5770	1871.1413
27.3240	3.0120	5.2272	746.6017
63.8067	3.9960	7.9879	4071.2922
33.0749	3.2100	5.7511	1093.9490
17.8302	2.6125	4.2226	317.9157
52.8806	3.7535	7.2719	2796.3540
10.4455	2.1860	3.2319	109.1074
52.9102	3.7542	7.2739	2799.4868
53.4687	3.7673	7.3122	2858.8984
27.4348	3.0160	5.2378	752.6694
72.4945	4.1697	8.5144	5255.4457
87.0521	4.4319	9.3302	7578.0669
26.2632	2.9725	5.1248	689.7564
67.3977	4.0696	8.2096	4542.4554

41.0319	3.4491	6.4056	1683.6209
66.9836	4.0612	8.1844	4486.8051
66.7037	4.0556	8.1672	4449.3857
53.0318	3.7570	7.2823	2812.3766
45.9308	3.5812	6.7772	2109.6372
48.5476	3.6480	6.9676	2356.8722
24.0247	2.8855	4.9015	577.1868
32.4351	3.1891	5.6952	1052.0381
6.5779	1.8737	2.5648	43.2694
73.4827	4.1885	8.5722	5399.7020
72.8259	4.1760	8.5338	5303.6147
27.1515	3.0056	5.2107	737.2029
81.3376	4.3328	9.0187	6615.7986
31.7930	3.1679	5.6385	1010.7955
50.7137	3.7015	7.1214	2571.8815
20.7466	2.7478	4.5548	430.4225
37.1548	3.3369	6.0955	1380.4760
41.5044	3.4623	6.4424	1722.6120
43.8410	3.5261	6.6213	1922.0300
25.9752	2.9616	5.0966	674.7114
45.3554	3.5662	6.7346	2057.1105
23.1440	2.8498	4.8108	535.6445
45.8359	3.5788	6.7702	2100.9287
25.3181	2.9364	5.0317	641.0046
118.7708	4.9155	10.8982	14106.5071
72.0813	4.1617	8.4901	5195.7196
25.2005	2.9318	5.0200	635.0648
71.6970	4.1543	8.4674	5140.4625
3.4805	1.5155	1.8656	12.1141
54.0257	3.7804	7.3502	2918.7790
74.0621	4.1995	8.6059	5485.1880
25.3986	2.9395	5.0397	645.0886
22.3626	2.8174	4.7289	500.0880
45.4771	3.5694	6.7437	2068.1697
49.9669	3.6832	7.0687	2496.6869
22.2560	2.8129	4.7176	495.3274
38.6632	3.3814	6.2180	1494.8393
72.7222	4.1740	8.5277	5288.5241
13.1508	2.3604	3.6264	172.9423
105.2290	4.7211	10.2581	11073.1432
24.3774	2.8995	4.9373	594.2570
25.0249	2.9250	5.0025	626.2456
41.2215	3.4544	6.4204	1699.2144
61.9847	3.9576	7.8730	3842.1038

82.5800	4.3547	9.0874	6819.4609
99.8930	4.6399	9.9946	9978.6015
59.6900	3.9081	7.7259	3562.8945
18.1621	2.6286	4.2617	329.8619
41.5818	3.4645	6.4484	1729.0486
126.4954	5.0199	11.2470	16001.0931
29.1662	3.0782	5.4006	850.6700
37.1024	3.3353	6.0912	1376.5900
45.5428	3.5711	6.7485	2074.1458
37.1614	3.3371	6.0960	1380.9679
35.0318	3.2721	5.9188	1227.2269
65.0894	4.0226	8.0678	4236.6356
39.2784	3.3993	6.2672	1542.7909
60.9410	3.9352	7.8065	3713.7999
121.3350	4.9506	11.0152	14722.1782
52.8649	3.7531	7.2708	2794.6928
56.3575	3.8340	7.5072	3176.1692
121.2133	4.9490	11.0097	14692.6584
50.7673	3.7028	7.1251	2577.3211
48.8847	3.6564	6.9918	2389.7169
76.7863	4.2504	8.7628	5896.1308
19.0095	2.6688	4.3600	361.3598
47.4693	3.6208	6.8898	2253.3319

4.1 Analysis of Data

Descriptive Statistics of the Gamma Distribution Data

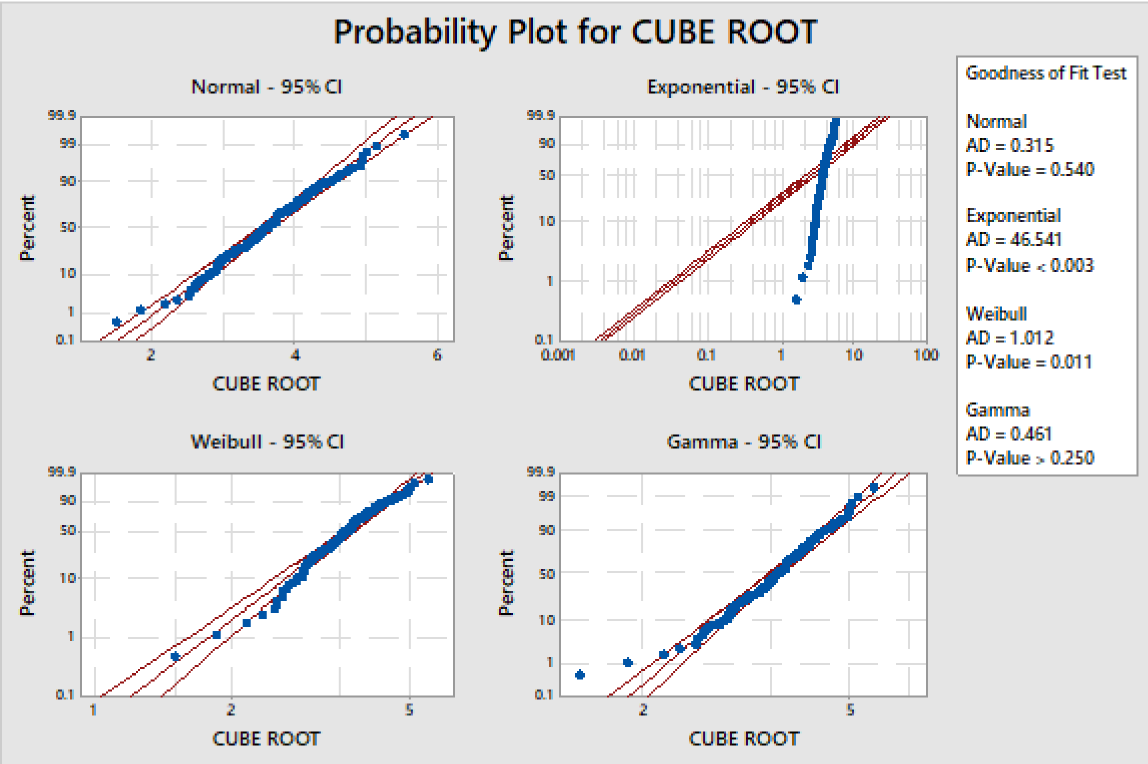
Mean	52.03
Standard Error	2.30
Median	46.70
Mode	#N/A
Standard Deviation	28.14
Sample Variance	791.64
Kurtosis	1.97
Skewness	1.20
Range	165.75
Minimum	3.48
Maximum	169.23
Sum	7804.36
Count	150.00

Distribution Identification for Cube Root
Descriptive Statistics

N	N*	Mean	StDev	Median	Minimum	Maximum	Skewness	Kurtosis
150	0	3.6155	0.6647	3.6010	1.5155	5.5313	0.0285	0.4180

Goodness of Fit Test

Distribution	AD	P
Normal	0.315	0.540
Exponential	46.541	<0.003
Weibull	1.012	0.011
Gamma	0.461	>0.250



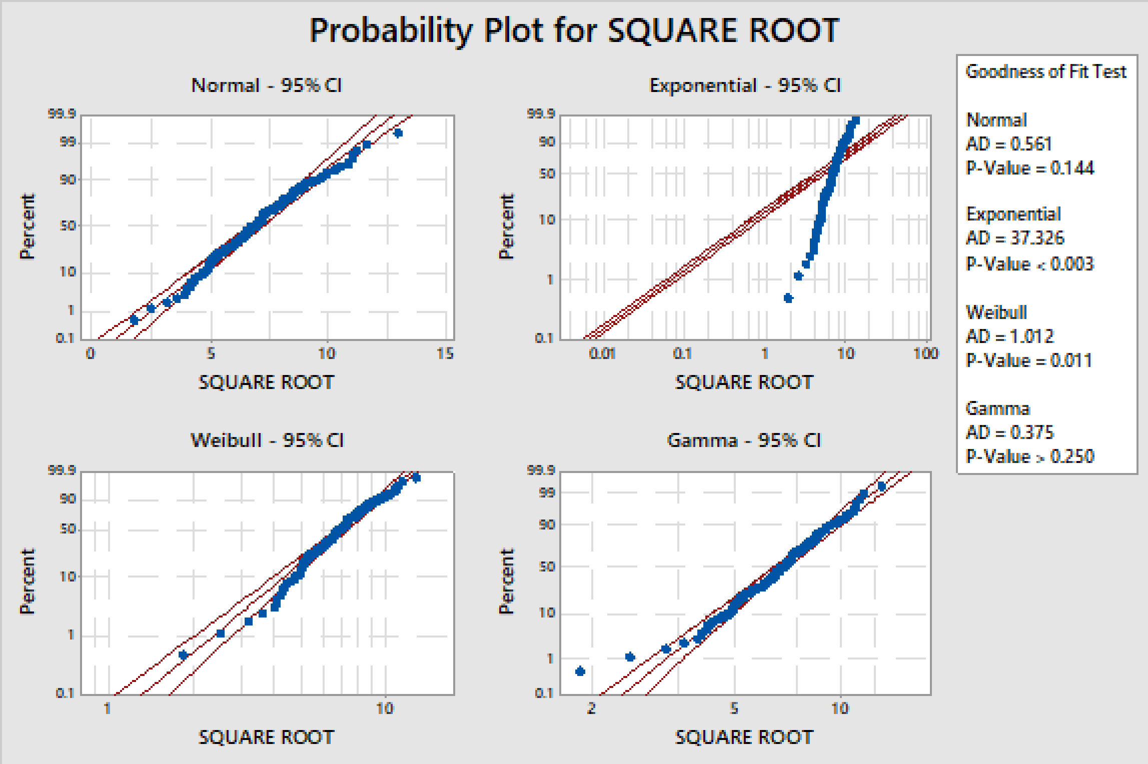
Interpretation: The probability plot of the cube root shows that the Gamma distribution with p-value = 0.250 is the best distribution that fits the data.

Distribution Identification for Square Root
Descriptive Statistics

N	N*	Mean	StDev	Median	Minimum	Maximum	Skewness	Kurtosis
150	0	6.9618	1.8939	6.8335	1.8656	13.0089	0.3561	0.3848

Goodness of Fit Test

Distribution	AD	P
Normal	0.561	0.144
Exponential	37.326	<0.003
Weibull	1.012	0.011
Gamma	0.375	>0.250



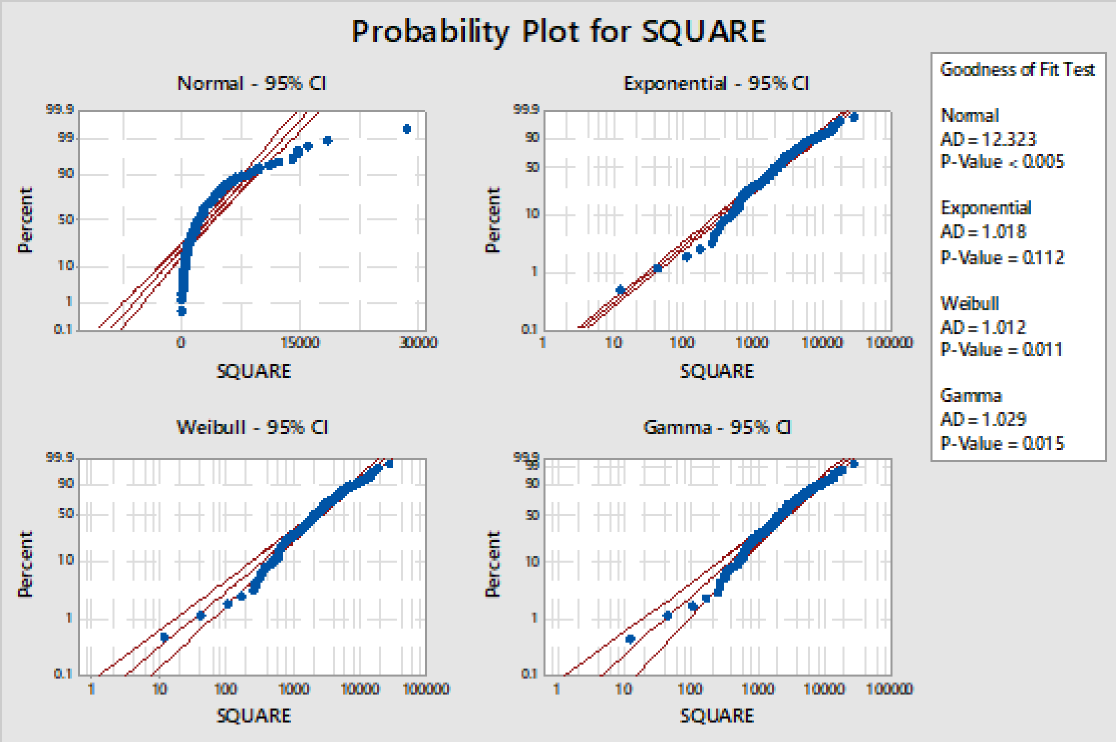
Interpretation: The probability plot for square root shows that the Gamma distribution approximates the data best than the other distributions. P-Value = 0.250.

Distribution Identification for Square
Descriptive Statistics

N	N*	Mean	StDev	Median	Minimum	Maximum	Skewness	Kurtosis
150	0	3493.39	4060.56	2181.48	12.1141	28638.9	2.8515	11.1845

Goodness of Fit Test

Distribution	AD	P
Normal	12.323	<0.005
Exponential	1.018	0.112
Weibull	1.012	0.011
Gamma	1.029	0.015



Interpretation: The probability plot for square shows that the exponential distribution fits the data best with p-value = 0.112.

4.2 Proof of the Transformations:

$$f(x)=\int_0^\infty \frac{\ell^{-\frac{x}{\beta}}x^{\alpha-1}}{\Gamma\alpha\beta^\alpha}dx \quad =\frac{1}{\Gamma\alpha\beta^\alpha}\int_0^\infty \ell^{-\frac{x}{\beta}}x^{\alpha-1}dx$$

Let $v=\frac{x}{\beta}\Rightarrow x=v\beta\Rightarrow dx=\beta dv$

$$=\frac{1}{\Gamma\alpha\beta^\alpha}\int_0^\infty \ell^{-v}(v\beta)^{\alpha-1}\beta dv \quad => \quad \frac{1}{\Gamma\alpha\beta^\alpha}\int_0^\infty \ell^{-v}v^{\alpha-1}\beta^{\alpha-1}\beta^1 dv$$

$$=\frac{1}{\Gamma\alpha}\int_0^\infty \ell^{-v}v^{\alpha-1}dv \quad => \frac{1}{\Gamma\alpha}.\Gamma\alpha=\frac{1}{1}=1$$

Cubic Transformation

$x=y^3$

$$\Gamma(\alpha,\theta)=\frac{1}{\Gamma(\alpha)\theta\alpha}\int_0^\infty x^{\alpha-1}\ell^{-\frac{x}{\theta}}dx \quad . \quad . \quad . \quad (1)$$

Using Jacobiam Transformation, we have that $dx=3y^2dy$

Inserting in equation (1)

$$\Gamma(\alpha,\theta)=\frac{1}{\Gamma(\alpha)\theta^\alpha}\int_0^\infty y^{3(\alpha-1)}\ell^{-\frac{y^3}{\theta}}3y^2dy \quad . \quad . \quad . \quad (2)$$

$$\Gamma(\alpha,\theta)=\frac{3}{\Gamma(\alpha)\theta^\alpha}\int_0^\alpha y^{3\alpha}.y^{-3}\ell^{-\frac{y^3}{\theta}}y^2dy \quad . \quad . \quad . \quad (3)$$

$$\Rightarrow \Gamma(\alpha,\theta) = \frac{3}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} y^{3\alpha} . y^{-1} \ell^{-\frac{y^3}{\theta}} dy \qquad . \qquad . \qquad . \qquad (4)$$

Again, Let $Z = \frac{y^3}{\theta}$ => $Z\theta = y^3$

$$y = Z^{\frac{1}{3}} . \theta^{\frac{1}{3}} \quad \Rightarrow \quad \frac{dy}{dz} = \frac{1}{3} Z^{\frac{-2}{3}} \theta^{\frac{1}{3}}$$

$$dy = \frac{1}{3} Z^{\frac{-2}{3}} \theta^{\frac{1}{3}} dz$$

Insert in equation 4

$$\frac{3}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} y^{3\alpha} . y^{-1} \ell^{-\frac{y^3}{\theta}} \frac{1}{3} z^{\frac{-2}{3}} \theta^{\frac{1}{3}} dz \qquad . \qquad . \qquad . \qquad (5)$$

$$\frac{3}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} y^{3\alpha-1} \ell^{-\frac{y^3}{\theta}} \frac{1}{3} z^{\frac{-2}{3}} \theta^{\frac{1}{3}} dz \qquad \Rightarrow \frac{3}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} z^{\frac{1}{3}(3\alpha-1)} \theta^{\frac{1}{3}(3\alpha-1)} \ell^{-z} \frac{1}{3} z^{-\frac{2}{3}} \theta^{\frac{1}{3}} dz$$

$$\Rightarrow \frac{1}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} z^{\alpha} z^{-\frac{1}{3}} \theta^{\alpha} . \theta^{-\frac{1}{3}} \ell^{-z} z^{-\frac{2}{3}} \theta^{\frac{1}{3}} dz \qquad . \qquad . \qquad . \qquad (6)$$

$$\Rightarrow \frac{1}{\Gamma(\alpha)} \int_0^{\infty} z^{(\alpha-1)} \ell^{-z} dz \qquad . \qquad . \qquad . \qquad (7)$$

$$\Gamma(\alpha,\theta) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} z^{\alpha-1} \ell^{-z} dz = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1$$

This shows that the transformation is a probability density function.

Square Transformation

$x = y^2$

$$\Gamma(\alpha,\theta) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} x^{\alpha-1} \ell^{\frac{-x}{\theta}} dx \qquad . \qquad . \qquad . \qquad (1)$$

Using Jacobiam Transformation, we obtain $x = y^2$

$$\frac{dx}{dy} = 2y \Rightarrow dx = 2ydy$$

Insert y in equation 1

$$\Gamma(\alpha,\theta) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} y^{2(\alpha-1)} \ell^{-\frac{y^2}{\theta}} 2ydy \qquad . \qquad . \qquad . \qquad (2)$$

$$\Gamma(\alpha,\theta) = \frac{2}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} y^{2\alpha} y^{-2} \ell^{-\frac{y^2}{\theta}} y^{1} dy \qquad . \qquad . \qquad . \qquad (3)$$

$$\Rightarrow \Gamma(\alpha,\theta) = \frac{2}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} y^{2\alpha} . y^{-1} \ell^{-\frac{y^2}{\theta}} dy \qquad . \qquad . \qquad . \qquad (4)$$

Hence, let $z = \frac{y^2}{\theta}$ => $y^2 = z\theta$

$$y = z^{\frac{1}{2}} \theta^{\frac{1}{2}} \quad \Rightarrow \frac{dy}{dz} = \frac{1}{2} z^{-\frac{1}{2}} \theta^{\frac{1}{2}} \quad while \; dy = \frac{1}{2} z^{-\frac{1}{2}} \theta^{\frac{1}{2}} dz$$

Insert *dy* in equation 4

$$\Rightarrow \Gamma(\alpha,\theta)=\frac{2}{\Gamma(\alpha)\theta^{\alpha}}\int_0^\infty y^{2\alpha}.y^{-1}\ell^{-\frac{y^2}{\theta}}\frac{1}{2}z^{-\frac{1}{2}}\theta^{\frac{1}{2}}dz$$

$$\Gamma(\alpha,\theta)=\frac{2}{\Gamma(\alpha)\theta^{\alpha}}\int_0^\infty\left(z^{\frac{1}{2}}\theta^{\frac{1}{2}}\right)^{2\alpha-1}\ell^{-\frac{z\theta}{\theta}}\frac{1}{2}z^{-\frac{1}{2}}\theta^{\frac{1}{2}}dz \quad . \quad . \quad . \quad (5)$$

$$\Rightarrow \frac{1}{\Gamma(\alpha)\theta^{\alpha}}\int_0^\alpha z^{\alpha}.z^{-\frac{1}{2}}\theta^{\alpha}\theta^{-\frac{1}{2}}\ell^{-z}z^{-\frac{1}{2}}\theta^{\frac{1}{2}}dz \quad . \quad . \quad . \quad (6)$$

$$\Rightarrow \Gamma(\alpha,\theta)=\frac{1}{\Gamma(\alpha)}\int_0^\infty z^{\alpha}z^{-1}\ell^{-z}dz \quad . \quad . \quad . \quad (7)$$

$$\Rightarrow \Gamma(\alpha,\theta)=\frac{1}{\Gamma(\alpha)}\int_0^\infty z^{\alpha-1}\ell^{-z}dz=1$$

This shows that the transformation is a probability density function.

Square Root Transformation

$$x=y^{\frac{1}{2}}$$

$$\Gamma(\alpha,\theta)=\frac{1}{\Gamma(\alpha)\theta^{\alpha}}\int_0^\infty x^{\alpha-1}\ell^{\frac{-x}{\theta}}dx \quad . \quad . \quad . \quad (1)$$

If $x=y^{\frac{1}{2}}$ and using Jacobiam Transformation $\frac{dx}{dy}=\frac{1}{2}y^{-\frac{1}{2}}$ and $dx=\frac{1}{2}y^{-\frac{1}{2}}dy$

Insert dx in equation 1

$$\Gamma(\alpha,\theta)=\frac{1}{\Gamma(\alpha)\theta^{\alpha}}\int_0^\infty y^{\frac{1}{2}(\alpha-1)}\ell^{-\frac{y^{\frac{1}{2}}}{\theta}}\frac{1}{2}y^{-\frac{1}{2}}dy \quad . \quad . \quad . \quad (2)$$

$$\Gamma(\alpha,\theta)=\frac{1}{2\Gamma(\alpha)\theta^{\alpha}}\int_0^\infty y^{\frac{\alpha}{2}}y^{\frac{-1}{2}}\ell^{-\frac{y^{\frac{1}{2}}}{\theta}}y^{-\frac{1}{2}}dy \quad . \quad . \quad . \quad (3)$$

$$\Rightarrow \Gamma(\alpha,\theta)=\frac{1}{2\Gamma(\alpha)\theta^{\alpha}}\int_0^\infty y^{\frac{\alpha}{2}}y^{-1}\ell^{-\frac{y^{\frac{1}{2}}}{\theta}}dy \quad . \quad . \quad . \quad (4)$$

Therefore let $z=\frac{y^{1/2}}{\theta}\quad\Rightarrow z\theta=y^{\frac{1}{2}}$ *while* $y=z^2\theta^2$

Using Jacobian Transformation

$$\frac{dy}{dz}=2z\theta^2\quad\Rightarrow dy=2z\theta^2dz$$

Insert dy in equation 4

$$\Rightarrow \Gamma(\alpha,\theta)=\frac{1}{2\Gamma(\alpha)\theta^{\alpha}}\int_0^\infty\left(z^2\theta^2\right)^{\frac{\alpha}{2}}\left(z^2\theta^2\right)^{-1}\ell^{-z}2z^1\theta^2dz \quad . \quad . \quad (5)$$

$$\Rightarrow \Gamma(\alpha,\theta)=\frac{1}{2\Gamma(\alpha)\theta^{\alpha}}\int_0^\infty z^{\frac{2\alpha}{2}}\theta^{\frac{2\alpha}{2}}z^{-2}\theta^{-2}\ell^{-z}2z^1\theta^2dz \quad . \quad . \quad (6)$$

$$\Rightarrow \Gamma(\alpha,\theta)=\frac{1}{\Gamma(\alpha)\theta^{\alpha}}\int_0^\infty z^{\alpha}\theta^{\alpha}z^{-2}\ell^{-z}z^1dz$$

$$\Rightarrow \Gamma(\alpha,\theta)=\frac{1}{\Gamma(\alpha)}\int_0^\infty z^\alpha z^{-2}\ell^{-z}z^1 dz \quad . \quad . \quad . \quad (7)$$

$$\Rightarrow \Gamma(\alpha,\theta)=\frac{1}{\Gamma(\alpha)}\int_0^\infty z^{\alpha-1}\ell^{-z} dz \quad =1$$

This shows that the transformation is a probability density function.

5.0 SUMMARY

In this work, the distribution of weilbull distribution under cubic, square and square root transformation using simulated data was carried out. It was discovered that the mean and variance of cubic and square root transformation are normally distributed while the mean of square before and after transformation are unity; the variance of the transformed is greater than the untransformed distribution. Based on the findings, there should be caution in applying square root transformation to a data set where the multiplicative error model is a suitable representation and were the error component has a Weibull distribution. This implies the transformation increases the error variance. We also discovered that using cubic and square root transformation in solving gamma distribution is normally fitted while square transformation is not, hence cubic and square root transformation is preferable that square transformation for a normal analysis.

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